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PHYSICS AND MATHEMATICS

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NOTES ON MAGNETO-HYDRODYNAMICS

VI

FLUID MAGNETIC EQUATIONS - GENERAL PROPERTIES

by

A. A. Blank and Harold Grad

July 1, 1958

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Institute of Mathematical Sciences

NEW YORK UNIVERSITY
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PREFACE

In this note we describe and collect for future reference in this series some general properties of the equations of fluid magnetics. The generality of the equations described in MH-1 makes mathematical analysis forbidding, and it is necessary to specialize for the purpose of analysis. In this supplement to MH-I we describe the fluid magnetic systems to which we shall have most frequent reference. Here, we shall adopt a set of standard non-dissipative fluid equations coupled with Maxwell's equations by a Lorentz force term and some form of Ohm's law. Within this frame, there is a considerable amount of flexibility; e.g., in the choice of an equation of state and of a form of Ohm's law, and in the decision to include or omit displacement current, electrostatic forces, and electromagnetic momentum. An important criterion in making these decisions is the mathematical tractability of the resultant equations; this desideratum should on occasion even override in importance the criterion of the precise physical validity of a given assumption.

The assistance of H. Kranzer in preparing this note is gratefully acknowledged.

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1. The Electromagnetic Equations

To describe the electromagnetic field we shall employ either the complete equations of Maxwell

$$(1.1)\quad \begin{cases} \dfrac{\partial B}{\partial t} + \mathrm{curl}\ E = 0, \quad \mathrm{div}\ B = 0 \\[2ex] -\dfrac{1}{c^2}\dfrac{\partial E}{\partial t} + \mathrm{curl}\ B = \mu J, \quad \mathrm{div}\ E = q/\varkappa \end{cases}$$

or, alternatively, the system in which displacement current is omitted,[1]

$$(1.2)\quad \begin{cases} \dfrac{\partial B}{\partial t} + \mathrm{curl}\ E = 0, \quad \mathrm{div}\ B = 0 \\[2ex] \mathrm{curl}\ B = \mu J, \quad \mathrm{div}\ E = q/\varkappa \end{cases}$$

This system will be called the pre-Maxwell equations since it is a formalization of the electromagnetic theory in existence before Maxwell's introduction of displacement current. Maxwell's equations are Lorentz invariant, but the system (1.2) is invariant under Galilean transformation with the quantities $E_u = E + u \times B$ (polar) and B (axial) treated as vectors. Since we shall wish to couple the electromagnetic equations with non-relativistic equations of the fluid, the system (1.2) is, in a sense, the more appropriate, but for the present the choice of electromagnetic equations is left open. From (1.1) we obtain the conventional form of the charge conservation equation

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[1] See MH-V, Theory of Maxwell's Equations without Displacement Current.

(1.3) $\frac{\partial q}{\partial t} + \text{div } J = 0$

but from (1.2) we have

(1.4) $\text{div } J = 0.$

Accordingly, the system (1.2) admits only solutions corresponding to the flow of currents in closed circuits; in particular, the singular case of a moving point charge cannot be considered as a localized singular current source.

The energy equation derived from (1.1) is

(1.5) $$\frac{\partial}{\partial t}\left(\frac{1}{2\mu} B^2 + \frac{1}{2} \chi E^2\right) + \text{div}\left(\frac{1}{\mu} E \times B\right) = - E \cdot J.$$

From (1.2) we obtain instead of (1.5) the relation

(1.6) $$\frac{\partial}{\partial t}\left(\frac{1}{2\mu} B^2\right) + \text{div}\left(\frac{1}{\mu} E \times B\right) = -E \cdot J,$$

and this will be taken as the formulation of energy conservation for the system. In other words, when we drop displacement current we treat magnetic energy alone; the significance of this will later be made clear.

The electromagnetic momentum equation may be obtained from (1.1) by use of the identity

(1.7) $$\text{curl } Z \times Z = \text{div}\left(Z^i Z^j - \frac{Z^2}{2} \delta^{ij}\right) + Z^i \text{div } Z$$

as follows:

(1.8) $$-\chi \frac{\partial}{\partial t}(E \times B) + \text{div } M = qE + J \times B$$

where the Maxwell stress tensor is defined by

$$(1.9)\quad \begin{cases} M = M_E + M_B \\ M_E^{ij} = \chi\left\{E^i E^j - \frac{1}{2} E^2 \delta^{ij}\right\} \\ M_B^{ij} = \frac{1}{\mu}\left\{B^i B^j - \frac{1}{2} B^2 \delta^{ij}\right\} \end{cases}$$

Using the identity (1.7) we obtain from the system (1.2) the equation

$$(1.10)\quad \operatorname{div} M_B = J \times B$$

which will be taken as the appropriate formulation of momentum conservation for the pre-Maxwell system. When we ignore displacement current, then, we choose to ignore electric forces as compared with magnetic forces as well as electric energy compared with magnetic energy. These stipulations will appear less arbitrary and even necessary when we consider the total energy and momentum equations including both the fluid and the electromagnetic contributions.

Integral form of the electromagnetic equations

We shall treat problems for which there may occur singular surfaces where we allow discontinuities in certain quantities. We shall therefore take the integral form of the equations as fundamental and derive discontinuity conditions from these. Maxwell's equations may be written in integral form as follows:

$$(1.11a)\quad \frac{d}{dt}\int_{\Sigma} B\cdot dS + \oint_{\Gamma} E\cdot dx = 0,$$

$$(1.11b)\quad \oint_{S} B\cdot dS = 0,$$

$$(1.11c)\quad -\frac{1}{c^2}\frac{d}{dt}\int_{\Sigma} E\cdot dS + \oint_{\Gamma} B\cdot dx = \mu\int_{\Sigma} J\cdot dS,$$

$$(1.11d)\quad \oint_{S} E\cdot dS = \frac{1}{\chi}\int_{D} q\, dV.$$

In these equations Γ stands for the boundary curve of a surface Σ, S for the boundary surface of a domain D. Throughout, we shall use the symbols D to denote a domain, S to denote a closed surface, Σ a surface with boundary, Γ a closed curve, C an open arc. For moving surfaces we have the alternative forms,

$$(1.12)\quad \frac{d}{dt}\int_{\Sigma(t)} B\cdot dS + \oint_{\Gamma(t)} (E + v\times B)\cdot dx = 0$$

$$(1.13)\quad -\frac{1}{c^2}\frac{d}{dt}\int_{\Sigma(t)} E\cdot dS + \oint_{\Gamma(t)} (B - \frac{1}{c^2} v\times E)\cdot dx$$

$$= \mu\int_{\Sigma(t)} (J - qv)\cdot dS$$

The moving surface $\Sigma(t)$ is to be thought of purely mathematically as a one-parameter family of surfaces given by equations $x = f(\alpha,\beta,t)$ with v defined as the derivative $\partial x/\partial t$ at a fixed point (α,β) of $\Sigma(t)$. No relativistic or galilean conceptions are involved. Since v enters the line integrals only in the combinations $v \times B \cdot dx$ and $v \times E \cdot dx$, we see that any component of v tangential to Γ is irrelevant. Similarly, only the component normal to the surface affects $qv \cdot dS$. In other words, the parametrization of Σ (or of Γ) which defines a "fixed point" moving with the surface is immaterial.

The integral forms of the energy and momentum equations (1.5), (1.6) and (1.8) taken on a stationary domain D are, respectively,

$$(1.14) \qquad \frac{d}{dt}\int_D \left(\frac{1}{2\mu} B^2 + \frac{1}{2}\chi E^2\right)dV + \oint_S \frac{1}{\mu}(E \times B)\cdot dS = -\int_D E\cdot J \, dV,$$

$$(1.15) \qquad \frac{d}{dt}\int_D \frac{1}{2\mu} B^2 dV + \oint_S \frac{1}{\mu}(E \times B)\cdot dS = -\int_D E\cdot J \, dV,$$

$$(1.16) \qquad -\chi\frac{d}{dt}\int_D (E \times B)^i dV + \oint_S M^{ij} dS_j = \int_D (qE + J \times B)^i dV.$$

Jump conditions at a discontinuity surface

Jump conditions across a discontinuity surface can be derived from the various integral forms. Although the same results are suggested by the differential equations they are not derivable from them. The integral formulation is basic here, leading to the differential equations under assumptions of smoothness and to jump conditions under other assumptions. The precise nature of the jump condition depends upon the choice of the variables assumed to be bounded.

Let the subscripts n and t designate respectively components of a vector normal and tangential to the discontinuity surface. From (1.11b) we obtain, under the assumption that B is bounded,

$$[B_n] = 0, \tag{1.17}$$

i.e., that the normal component of B is continuous across a discontinuity surface. From (1.11a), under the assumption that B and $\partial B/\partial t$ are bounded, we find

$$[E_t] = 0, \tag{1.18}$$

namely, that the tangential component of E is continuous across a stationary discontinuity surface. Discontinuities of E_t may occur at isolated instants, however. For example, in a conducting fluid if E_t is assumed to be discontinuous at the initial instant, a shock wave results which immediately resolves the discontinuity (of course, $\partial B/\partial t$ is initially infinite). Practically, an instantaneous discontinuity in E_t may be produced by the head-on collision of two shocks. For a moving discontinuity surface we get a more general condition from (1.12),

$$[(E + v \times B)_t] = 0. \tag{1.19}$$

The inhomogeneous equations (1.11c) and (1.11d) require a somewhat different treatment for q and J. If discontinuities in

E and B are allowed then we must allow infinities in q and J. Specifically, we obtain

$$(1.20a) \qquad n \times [B_t] = n \times [B] = \mu j$$

$$(1.20b) \qquad [E_n] = Q/\mathcal{K}$$

where j denotes a surface current and Q a surface charge density. The positive normal has been taken in the direction from side (0) to side (1) where the jump in a quantity Z is defined by $[Z] = Z_1 - Z_o$.

The energy and momentum equations do not yield independent jump conditions, but these conditions are also of interest. In (1.20a) we take the scalar product with E_t (which is continuous across a stationary interface) and find

$$(1.21) \qquad E \cdot j = [\frac{1}{\mu} B] \times E \cdot n.$$

The left side in (1.21) has a plain interpretation as an energy rate while the right side is associated with the Poynting flux.

Example of the current-carrying loop

In illustration, we apply these formulas to the simple case of a single loop of current-carrying wire of finite thickness. For simplicity, we drop the displacement current. Let D denote the entire domain consisting of the wire D_1, and the vacuum domain D_2, and let S denote the common boundary surface of D_1 and D_2. Using the regularity condition that the total electromagnetic flux must be negligible in the neighborhood of infinity, we have the rate of change of the total magnetic energy from

$$(1.22) \qquad \frac{d}{dt}\int_D \frac{1}{2\mu} B^2 dV = -\int_{D_1} E\cdot J\, dV,$$

and, for the magnetic energy in the wire and vacuum, respectively,

$$(1.23) \qquad \frac{d}{dt}\int_{D_2} \frac{1}{2\mu} B^2 dV = \oint_S \frac{1}{\mu} E\times B\cdot dS$$

$$(1.24) \qquad \frac{d}{dt}\int_{D_1} \frac{1}{2\mu} B^2 dV = -\int_{D_1} E\cdot J\, dV - \oint_S \frac{1}{\mu} E\times B\cdot dS.$$

Here, the positive normal to S points into the vacuum domain. Insofar as E and J can be replaced by their average values on a cross-section of the wire, we may approximate the right side in (1.22) by setting

$$(1.25) \qquad \int_{D_1} E\cdot J\, dV = I\oint_\Gamma E\cdot dx$$

where I denotes the total current through a cross-section of the wire (and is independent of the choice of cross-section) and $\oint_C E\cdot dx$ is the e.m.f. around the wire. This is an electrical circuit approximation. If the wire is thin and the current is not too strongly localized within a cross-section, then we may neglect the magnetic energy within the wire compared to that in the vacuum. We shall then have in approximation,

$$\int_{D_1} E\cdot J \, dV = -\oint_{S} E\times B\cdot dS = I \oint_{\Gamma} E\cdot dx$$

Alternatively, if we assume that curl E = 0 in the wire, that is, that B is not changing within the wire, then using div J = 0 we have from MH-V(6.5)*

$$\int_{D_1} E\cdot J \, dV = I \oint_{\Gamma} E\cdot dx$$

no matter what the shape or size of the wire. We may also utilize the slightly weaker assumption that E_t is a local surface gradient on the surface S of the wire, and obtain once more from MH-V(6.19) that

$$\oint_{S} \frac{1}{\mu} E\times B\cdot dS = -\oint_{\Gamma} E\cdot dx \int_{\tilde{\Gamma}} \frac{1}{\mu} B\cdot dx = -I \oint_{\Gamma} E\cdot dx$$

where $\tilde{\Gamma}$ is a closed curve bounding a cross-section of the wire.

In the absence of displacement current, we see that the field concept of Poynting energy flux across the surface of the wire is closely related to the familiar elementary circuit quantity, voltage times current.

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*Refers to Notes on Magneto-Hydrodynamics, Number V, equation number (6.5).

2. The Fluid Equations

From MH-I (7) we take the fluid conservation equations in Lagrangian form[2], integrating over a domain D(t) which is carried along by the fluid with velocity u.

$$(2.1) \qquad \frac{d}{dt}\int_{D(t)} \rho dV = 0$$

$$(2.2) \qquad \frac{d}{dt}\int_{D(t)} \rho u^i dV + \oint_{S(t)} P^{ij} dS_j = \int_{D(t)} X^i dV$$

$$(2.3) \qquad \frac{d}{dt}\int_{D(t)} \rho(\hat{e} + \frac{1}{2} u^2) dV + \oint_{S(t)} (u^i P^{ij} + Q^j) dS_j = \int_{D(t)} \mathcal{E}\, dV.$$

The quantity X^i denotes the force density, $\mathcal{E}$ the energy source, and Q^j heat flow. If we utilize Maxwell's equations (1.1) for the electromagnetic field, we shall specify

$$(2.4a) \qquad X = qE + J \times B + \rho g$$

$$(2.4b) \qquad \mathcal{E} = E \cdot J + \rho u \cdot g$$

where g can be interpreted as a gravitational field or any other conservative force field. The stress tensor P^{ij} will usually be assumed to be scalar,

$$(2.5) \qquad P^{ij} = p\delta^{ij},$$ and the heat flow term is dropped.

The quantity $\hat{e}$ denotes the specific internal energy or internal energy per unit mass. The equations (2.1) - (2.3) are assumed to be supplemented by an equation of state connecting the density ρ, pressure p, and internal energy $\hat{e}$. In these equations we have omitted dissipative terms such as viscosity and heat conduction.

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[2] For a relativistic treatment, see Paul Reichel, Basic Notions of Relativistic Hydromagnetics, NYO-7697, New York University.

The conservation equations (2.1)-(2.3) may be given in differential form as follows:

$$(2.6a)\qquad \frac{d\rho}{dt} + \rho \operatorname{div} u = 0$$

$$(2.6b)\qquad \rho\frac{du}{dt} + \operatorname{grad} p = qE + J \times B + \rho g$$

$$(2.6c)\qquad \rho\frac{d\hat{e}}{dt} + p \operatorname{div} u = (E + u \times B)\cdot(J - qu),$$

where the Lagrangian time derivative d/dt is defined by

$$(2.7)\qquad \frac{d}{dt} = \frac{\partial}{\partial t} + (u\cdot\nabla)$$

and all the simplifying assumptions mentioned above have been employed. The derivation of these equations from the integral forms involves an amount of elementary manipulation. An alternative form of (2.6c) in which entropy replaces internal energy as the variable is often useful. We use the specific entropy $\hat{\eta}$ defined by

$$(2.8)\qquad T d\hat{\eta} = d\hat{e} + p d(1/\rho)$$

to obtain

$$(2.9\qquad \rho\frac{d\hat{\eta}}{dt} = \frac{1}{T}(E + u \times B)\cdot(J - qu).$$

In most of the later applications we shall not only omit the dissipative terms of fluid dynamics but it will be assumed that there is no electromagnetic dissipation either; we only consider it here for purposes of illustration. In other words, we shall generally assume the flow to be adiabatic and replace the energy equation by the condition that entropy is constant following a particle

$$(2.10) \qquad \frac{d\hat{\eta}}{dt} = 0.$$

The total conservation equations

For a fixed domain and no external force field (g = 0) the equations (2.2) and (2.3) take the form (cf. MH-I(6))

$$(2.11a) \qquad \frac{d}{dt}\int_D \rho u^i dV + \oint_S (\rho u^i u^j + p\delta^{ij}) dS_j = \int_D (qE + J \times B)^i dV$$

$$(2.11b) \qquad \frac{d}{dt}\int_D \rho(\hat{e} + \tfrac{1}{2} u^2) dV + \oint_S \left\{\rho(\hat{e} + \tfrac{1}{2} u^2) + p\right\} u \cdot dS$$

$$= \int_D E \cdot J \, dV$$

In contrast to the mass conservation equation, these equations should more properly be called "lack of conservation" equations. For example, the rate of change of momentum is given by a surface flow plus a distributed volume contribution. However, on combining the corresponding fluid and electromagnetic equations we should obtain a true conservation equation without the volume terms. In the case of energy, this condition is satisfied whether we combine (2.11b) with the Maxwell formula (1.14) or the pre-Maxwell formula (1.15) which omits displacement current. For momentum we obtain conservation only if we drop or retain the force term qE in (1.16) together with displacement current. Independently of this criterion, it is easy to see that in anything but a very poor conductor, qE is always small compared to J × B. Nevertheless, if displacement current is kept, then, whatever the physical importance of the force qE, it should be retained in the equations for mathematical soundness. For the same reason when displacement current is dropped then the term qE must also be dropped.

In summary, together with the Galilean invariant pre-Maxwell equations, (1.2), one should take the force equation

$$(2.12) \qquad \rho \frac{du}{dt} + \text{grad } p = J \times B$$

and either the energy equation

$$(2.13) \qquad \rho \frac{d\hat{e}}{dt} + p \text{ div } u = (E + u \times B) \cdot J$$

or the entropy equation

$$(2.14) \qquad \rho \frac{d\hat{\eta}}{dt} = \frac{1}{T}(E + u \times B) \cdot J.$$

The total conservation equations are

$$(2.15) \qquad \frac{d}{dt} \int_D \rho u^i dV + \oint_S \left\{ \rho u^i u^j + p\delta^{ij} - M_B^{ij} \right\} dS_j = 0$$

$$(2.16) \qquad \frac{d}{dt} \int_D \left\{ \rho(\hat{e} + \frac{1}{2}u^2) + \frac{1}{2\mu} B^2 \right\} dV$$

$$+ \oint_S \left\{ \rho(\hat{e} + \frac{1}{2}u^2)u^j + pu^j + \frac{1}{\mu} (E \times B)^j \right\} \cdot dS_j = 0$$

With the full Maxwell equations (1.1), relativistic fluid equations would be more suitable, but as a compromise, one might take the force equation (2.6b) and either the energy equation (2.6c) or the entropy equation (2.9). The total conservation equations are

$$(2.17)\qquad \frac{d}{dt}\int_D \left\{\rho u^i + \varkappa(E\times B)^i\right\} dV + \int_S \left\{\rho u^i u^j + p\delta^{ij} - M^{ij}\right\} dS_j = 0,$$

and

$$(2.18)\qquad \frac{d}{dt}\int \left\{(\rho\hat{e} + \tfrac{1}{2}u^2) + \frac{1}{2\mu}B^2 + \frac{\varkappa}{2}E^2\right\} dV$$

$$+ \oint_S \left\{\rho(\hat{e} + \tfrac{1}{2}u^2)u^j + pu^j + \frac{1}{\mu}(E\times B)^j\right\} dS_j = 0.$$

It is worth noting that in the case with no displacement current, q occurs only in the equation $q = \varkappa \,\mathrm{div}\, E$; this relation serves to determine q after E (and all other variables) have been determined by other conditions. Conservation of charge will in general be violated.

It is customary to think of the Poynting flux as the energy passing through the surface by radiation alone, but it actually represents the entire flux of field energy. We observe, for example, that the appearance of the Poynting flux in the pre-Maxwell equation (2.16) is not associated with a transfer of radiated energy.

Jump conditions at discontinuity surfaces

The jump condition derived from the mass conservation equation (2.1) for a stationary discontinuity surface is

(2.19) $$[\rho u_n] = 0.$$

If there is flow through the interface, $u_n \neq 0$, we have a shock wave. Shock discontinuities are considered in detail in MH-VIII. For present purposes, we restrict ourselves to discontinuities with $u_n = 0$ (Helmholtz discontinuity).

From the momentum conservation equation (2.15) (without displacement current) and from (1.17) we obtain the jump condition

$$[p]n^i - \frac{1}{\mu}[B^i]B_n + \frac{1}{2\mu}[B^2]n^i = 0.$$

If $B_n \neq 0$ it follows that there is no jump in these variables. If $B_n = 0$, we have

(2.20) $$[p + \frac{1}{2\mu} B^2] = 0, \qquad (B_n = 0).$$

Exactly the same result holds for a moving discontinuity surface. This result depends upon the assumption of scalar pressures. In this case, the magnetic force across a discontinuity surface is balanced by a pressure force which is normal to the surface. The Maxwell stress is normal only when B is either tangential or normal to the surface, and in the latter case, B is continuous. There is one important situation in which the assumption of scalar pressure can be dispensed with. This is the case of a current-carrying discontinuity surface which separates a fluid region where B vanishes, from a vacuum magnetic field (cf. MH-X). In that case the pressure must be exerted normally and satisfies the pressure balance (2.20).

From the momentum equation (2.17) with displacement current we have, using (1.18)

$$[p]n^i = \frac{1}{\mu}[B^i]B_n + \frac{1}{2\mu}[B^2]n^i - \chi E_t^i[E_n] - \frac{\chi}{2}[E^2]n^i = 0.$$

It follows on taking the scalar product with n, that

$$(2.21a) \quad [p + \frac{1}{2\mu} B^2 - \frac{\chi}{2} E^2] = 0;$$

and taking the cross-product with n we have from (1.20)

$$(2.21b) \quad B_n(j \times n) - QE_t = 0.$$

At a perfect conductor (see the following section), we would have $E_t = 0$ and the relations (2.21) are identical with those without displacement current; namely, $[p + \frac{1}{2\mu} B^2] = 0$ and $B_n = 0$. The existence of a surface current j is incompatible with the concept of finite conductivity. Consequently, for a finitely conductive fluid, B must be continuous and we must have either E continuous or $E_t = 0$. In the event $E_t = 0$, then for the problem of determining the fields on one side of the discontinuity surface, we may as well assume the conductor on the other side to be "perfect."

3. Ohm's Law:[3]

If J is an unknown of a problem it is necessary to supplement the electromagnetic equations (1.1) or (1.2) with an additional vector equation which is generally a counterpart to Ohm's Law. For a perfect conductor, we take

$$(3.1) \qquad E + u \times B = 0;$$

in this case the fluid is non-dissipative (cf. (2.9)). More generally, we have (see MH-IV(12)),

$$(3.2) \qquad E_u = R_{\|} J_u^{\|} + \xi J_u \times B + R_{\perp} J_u^{\perp}$$

where $R_{\|}$ (parallel resistivity), $R_{\perp}$ (perpendicular resistivity), and ξ (Hall coefficient) are three phenomenological "constants" of the medium. The symbols E_u and J_u denote values measured in a coordinate system moving with the fluid velocity u, namely,

$$(3.3) \qquad \begin{cases} E_u = E + u \times B \\ J_u = J - qu \, ; \end{cases}$$

when displacement current is omitted we take $J_u = J$. The symbols $\|$ and $\perp$ denote components perpendicular and parallel to the B-field; that is, $J_u = J_u^{\|} + J_u^{\perp}$, where

$$(3.4) \qquad J_u^{\|} = \frac{B(B \cdot J_u)}{B^2}, \qquad J_u^{\perp} = \frac{B \times (J \times B)}{B^2} .$$

The dissipation (rate of entropy production) is defined by

---

[3] See MH-IV for a general discussion.

(3.5) $$E_u \cdot J_u = R_{\parallel}\,(J_u^{\parallel})^2 + R_{\perp}\,(J_u^{\perp})^2 .$$

The complete set of fluid magnetic equations for a perfectly conducting medium (3.1) form a mathematically tractable system.[6] the Hall effect in (3.2) usually makes for complication even though it is non-dissipative.

Ohm's law is usually used to eliminate E rather than J. In particular, we have for a perfect conductor

(3.6) $$\frac{\partial B}{\partial t} - \mathrm{curl}(u \times B) = 0$$

or, in integral form, (cf. (1.12)),

(3.7) $$\frac{d}{dt} \int_{\Sigma(t)} B \cdot dS = 0.$$

This result means that magnetic flux is "frozen" to an element of surface which moves with the medium. Within a perfect conductor magnetic lines are identifiable in the sense that the fluid motion carries magnetic lines into magnetic lines (sometimes it is necessary to be careful in fixing magnetic lines to a conducting medium; compare the example of MH-V, p. 129).

From the simple form of Ohm's law, $E_u = RJ_u$, we obtain, on neglecting displacement current,

(3.8) $$\frac{\partial B}{\partial t} - \mathrm{curl}(u \times B) = -\frac{1}{\mu}\,\mathrm{curl}(R\ \mathrm{curl}\ B).$$

In addition to being carried with a surface element the magnetic flux diffuses with a diffusivity R/μ.

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[6] cf. K. O. Friedrichs and H. Kranzer, Notes on Magneto-Hydrodynamics No. VIII, NYO-6486, Nonlinear Wave Motion.

A simpler form of this diffusion equation is obtained in a two-dimensional geometry for which B has a z-component only, u has only x and y components, and no variables depend on z.[7] Interpreting B as a scalar, and using (3.2), we find

$$(3.9) \qquad \frac{\partial B}{\partial t} + \operatorname{div}(uB) = \frac{1}{\mu} \operatorname{div}(R_{\perp} \operatorname{grad} B).$$

The parallel resistivity, $R_{\|}$ does not enter, nor does ξ if it is a constant.

When displacement current is dropped, Ohm's law together with curl B = μJ allows one to eliminate all electromagnetic variables in terms of B. In addition to the fluid equations (which contain a term curl B × B in the momentum equation and something like $(\text{curl } B)^2$ in the energy equation), there is an equation such as (3.6) or (3.8), or a slightly more complicated equation derived from the general form of Ohm's law, (3.2).

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[7] The one-dimensional case is analyzed in NYO-7691, Marian H. Rose, On the Diffusion of a Conducting Fluid Across a Magnetic Field. The non-dissipative problem is treated in Notes on Magneto-Hydrodynamics VII, Fluid Dynamical Analogies by A. A. Blank and Harold Grad.

4. Characteristics

We have considered the two systems of non-dissipative fluid magnetic equations: a galilean system which combines classical equations of the fluid with the non-relativistic pre-Maxwell equations, and a hybrid system which combines the non-relativistic fluid equations with the Lorentz-invariant field equations of Maxwell. In order to evaluate the two systems we shall compare them to the complete relativistic system as a standard.

We have already seen that for the two systems under study the conservation principles take somewhat different forms. More refined distinctions can be made by studying the characteristics of the systems, that is, by examination of the propagation of weak disturbances. These systems and the relativistic system are all symmetric hyperbolic.[4] The existence theory and general properties of such systems have been extensively developed.

In each of these systems the

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[4] For the galilean system, see K. O. Friedrichs, Nonlinear Wave Motion in Magneto-hydrodynamics, LAMS-2105. K. O. Friedrichs and H. Kranzer, Nonlinear Wave Motion, Notes on Magnetohydrodynamics VIII, NYO-6486, New York University, Institute of Mathematical Sciences, (1958).

For the relativistic system, see P. Reichel, Basic Notions of Relativistic Hydromagnetics, NYO-7697, New York University, Institute of Mathematical Sciences, (1958).

Also, B. Zumino, Some questions in relativistic hydromagnetics, Phys. Rev. 108, No. 5, 1116-1121 (1957).

characteristic equation, which has as its roots the characteristic velocities at which weak signals propagate, is a polynomial of sixth degree. Letting V denote the characteristic velocity as measured in a frame at rest with respect to the fluid we obtain for the characteristic equation in the non-relativistic case,

$$(4.1) \qquad [\mu\rho V^2 - B_n^2][\mu^2\rho^2V^4 - \mu\rho V^2(\mu\rho a^2 + B^2) + \mu\rho a^2B_n^2] = 0$$

and in the relativistic case,

$$(4.2) \qquad [V^2(\mu\rho^* + \frac{B^2}{c^2}) - B_n^2][V^4\mu^2\rho^*(\rho^* + \frac{B^2}{c^2}) - V^2\left\{\mu\rho^*(\mu\rho a^2 + B^2) + \mu\rho\frac{a^2B_n^2}{c^2}\right\} + \mu\rho a^2B_n^2\ ] = 0.$$

In these equations n denotes the wave normal, $c^2 = (\mu\,\mathcal{K})^{-1}$ is the squared light speed and $\rho^* = \rho + \frac{1}{c^2}(p + \rho\hat{e})$ where ρ is the rest-mass density (rest-mass is conserved), and $\hat{e}$ represents the internal energy per unit rest mass measured in the frame at rest with respect to the fluid.

The principal difficulty with the non-relativistic system is that in the low density range, some of the characteristic speeds become unbounded. In fact, for the roots of (4.1) we have asymptotically in ρ:

$$(4.3) \qquad V^2 \sim \frac{a^2B_n^2}{B^2}\ ,\ \frac{B_n^2}{\mu\rho}\ ,\ \frac{B^2}{\mu\rho}\ .$$

In other words, for sufficiently low densities the non-relativistic equations permit the propagation of a signal in excess of light speed. For comparison we observe that the corresponding roots of

(4.2) are given by

$$(4.4) \qquad V^2 \sim \frac{a^2B_n^2}{B^2}, \quad \frac{B_n^2}{\mu\rho + B^2/c^2}, \quad \frac{B^2}{\mu\rho + B^2/c^2}.$$

From these observations it is clear in what realm the non-relativistic system is an adequate approximation to the relativistic system.

At this point, it is reasonable to ask whether the hybrid system has any advantages to compensate for the lack of invariance. In particular, to be useful, the hybrid system ought to be simpler than the relativistic system and at the same time provide a better approximation than the non-relativistic system in the low density limit. For the hybrid system, the propagation speeds are taken with respect to a stationary frame since the concept of velocity relative to the fluid has no invariant meaning. We use V_o to designate velocity relative to the given frame and, for comparison with the other systems, set $V = V_o - u_n$. The characteristic equation for the hybrid system takes the rather complicated form

$$(4.5) \qquad \Phi\Psi + \frac{1}{c^2}\left\{\alpha\Psi + (V_o^2 + u_3^2)\,\Phi\,\Omega\right\} + \frac{1}{c^4}\,V_o^2\,\Omega\,\left\{\alpha + u_3^2B_n^2\right\} = 0$$

where

$$\Phi = \mu\rho V^2 - B_n^2, \quad \Psi = \mu^2\rho^2V^2(V^2-a^2) - \Omega, \quad \Omega = \mu\rho(V^2B^2 - a^2B_n^2)$$

$$\alpha = B_n^2V_o^2 + (VB_t + u_2B_n)^2,$$

$$B_t^2 = B^2 - B_n^2, \quad u_2 = u\cdot(B-nB_n)/B_t, \quad u_3 = u\cdot(n\times B)/B_t.$$

The product $\Phi\Psi$ is the left side of equation (4.1). It is clear that this is a far more complicated equation than either of (4.1) and (4.2). In the zero density limit the characteristic speeds are given by

$$(4.6) \quad \left\{ \begin{array}{l} (V_o - u_n)^2 = V^2 = a^2 \dfrac{B_n^2}{B^2} , \\[2ex] V_o - u_n = V = - \dfrac{B_n}{|B|} \left\{ u_{\|} + \sqrt{c^2 - u_{\perp}^2} \right\} \\[2ex] V_o^2 = (V + u_n)^2 = c^2 \end{array} \right.$$

where $u_{\|} = \dfrac{u \cdot B}{|B|}$ and $u_{\perp}^2 = u^2 - u_{\|}^2$.

It is clear that the propagation speeds (in either the rest frame or the fluid frame) are not bounded by light speed c unless the fluid velocity is dropped as being small compared to c. The hybrid system is useful only when $1/c^2$ is considered small as well as ρ and, with that understanding, only the lowest order terms are pertinent. The relativistic correction to the galilean system provided by the hybrid system is applicable only in a limited range and at the cost of greater complexity than even the fully relativistic system. There would then seem to be no practical value in the use of the hybrid system.

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